LECTURE: 3-4 THE CHAIN RULE

When you have a function that is a composite function, like $y = \sqrt{x^2+1}$, the formulas we have so far do not let us find y'. However, if you write your composite function as $f \circ g$, we have a formula for the derivative.

The Chain Rule: If f and g are differentiable and $F = f \circ g$, then F is differentiable and

$$F'(x) = f'(g(x))g'(x).$$

If $y = f(u)$ and $u = g(x)$, let
$\Delta u = g(x+\Delta x) - g(x)$
$\Delta y = f(u+\Delta u) - f(u)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

problems → Δu may $= 0$...

Example 1: Write the composite function in the form $f(g(x))$ and then find y'.

(a) $y = (1+3x)^9$ $\quad f(x) = x^9,\ g(x) = 1+3x$

$y' = 9(1+3x)^8 \cdot \frac{d}{dx}(1+3x)$

$y' = 9(1+3x)^8 \cdot 3$

$\boxed{y' = 27(1+3x)^8}$

(b) $y = \dfrac{1}{(x^2+2x-5)^9} = (x^2+2x-5)^{-9}$ $\quad f(x) = x^{-9},\ g(x) = x^2+2x-5$

$y' = -9(x^2+2x-5)^{-10} \cdot \frac{d}{dx}(x^2+2x-5)$

$y' = \dfrac{-9}{(x^2+2x-5)^{10}} \cdot (2x+2)$

$\boxed{y' = \dfrac{-18(x+1)}{(x^2+2x-5)^{10}}}$

Example 2: Write the composite function in the form $f(g(x))$. Then, find y'.

(a) $y = \cos(x^3)$ $\quad f(x) = \cos x,\ g(x) = x^3$

$y' = -\sin(x^3) \cdot \frac{d}{dx}(x^3)$

$y' = -\sin(x^3) \cdot 3x^2$

$\boxed{y' = -3x^2 \sin(x^3)}$

(b) $y = \cos^3(x) = (\cos x)^3$ $\quad f(x) = x^3,\ g(x) = \cos x$

$y' = 3(\cos x)^2 \cdot \frac{d}{dx} \cos x$

$y' = 3\cos^2 x(-\sin x)$

$\boxed{y' = -3\cos^2 x \sin x}$

Chain + quotient rule!

Example 3: Find the derivative of $f(x) = \left(\dfrac{x+5}{2x-1}\right)^5$.

$$f'(x) = 5\left(\frac{x+5}{2x-1}\right)^4 \cdot \frac{d}{dx}\left(\frac{x+5}{2x-1}\right)$$

$$= 5\left(\frac{x+5}{2x-1}\right)^4 \cdot \left[\frac{(2x-1)(1)-(x+5)\cdot 2}{(2x-1)^2}\right]$$

$$= \frac{5(x+5)^4(2x-1-2x-10)}{(2x-1)^6}$$

$$= \frac{5(x+5)^4(-11)}{(2x-1)^6}$$

$$= \boxed{\frac{-55(x+5)^4}{(2x-1)^6}}$$

product, then chain...

Example 4: Find the derivative of $f(x) = (2x-1)^6(x^3-2x+1)^3$

$$f'(x) = \left[\frac{d}{dx}(2x-1)^6\right](x^3-2x+1)^3 + (2x-1)^6 \cdot \left[\frac{d}{dx}(x^3-2x+1)^3\right]$$

$$= 6(2x-1)^5 \cdot \underbrace{2}_{\text{chain bit}}(x^3-2x+1)^3 + (2x-1)^6 \cdot 3(x^3-2x+1)^2 \underbrace{(3x^2-2)}_{\text{chain bit}}$$

$$= 12(2x-1)^5(x^3-2x+1)^3 + 3(2x-1)^6(x^3-2x+1)^2(3x^2-2)$$

$$= \underbrace{3(2x-1)^5(x^3-2x+1)^2}_{\text{this is the greatest common factor}}\left(4(x^3-2x+1) + (2x-1)(3x^2-2)\right)$$

$$= 3(2x-1)^5(x^3-2x+1)^2(4x^3-8x+4+6x^3-3x^2-4x+2)$$

$$= \boxed{3(2x-1)^5(x^3-2x+1)^2(10x^3-3x^2-12x+6)}$$

Example 5: Find the derivative of the following functions.

(a) $y = e^{x \sec x}$

$$y' = e^{x\sec x} \cdot \frac{d}{dx}(x \sec x)$$

$$= e^{x\sec x}(1 \sec x + x \cdot \sec x \tan x)$$

$$= \boxed{\sec x\, e^{x \sec x}(1 + x\tan x)}$$

(b) $y = \underline{\sin}(\sin(\sin x))$

$$y' = \cos(\sin(\sin x))\frac{d}{dx}\sin(\sin x)$$

$$= \cos(\sin(\sin x))\cos(\sin x)\frac{d}{dx}\sin x$$

$$= \boxed{\cos(\sin(\sin x))\cos(\sin x)\cos x}$$

Review: The Chain Rule: If f and g are differentiable and $F = f \circ g$, then F is differentiable and

$$F'(x) = f'(g(x))\, g'(x)$$

TYPO - take out prime.

Example 6: Let $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g'(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(x) = f'(g(x))\, g'(x)$$
$$F'(5) = f'(g(5))\, g'(5)$$
$$= f'(-2)(6)$$
$$= 4 \cdot 6$$
$$= \boxed{24}$$

Example 7: Find the derivatie of the following functions.

(a) $g(x) = \sqrt[5]{x^3 - 1} = (x^3-1)^{1/5}$

$$g'(x) = \tfrac{1}{5}(x^3-1)^{1/5-1} \cdot \frac{d}{dx}(x^3-1)$$
$$= \tfrac{1}{5}(x^3-1)^{-4/5}(3x^2)$$
$$= \boxed{\frac{3x^2}{5(x^3-1)^{4/5}}}$$

(b) $h(x) = \sin^5(4x^2) = (\sin(4x^2))^5$

$$h'(x) = 5(\sin(4x^2))^4 \cdot \left(\frac{d}{dx}\sin(4x^2)\right)$$
$$= 5\sin^4(4x^2)\cos(4x^2) \cdot \frac{d}{dx}4x^2$$
$$= 5\sin^4(4x^2)\cos(4x^2) \cdot 8x$$
$$= \boxed{40x\sin^4(4x^2)\cos(4x^2)}$$

Formula: Derivative of $y = b^x$ $\boxed{\dfrac{d}{dx}(b^x) = (\ln b)b^x}$ ← memorize this.

Why: $y = b^x = (e^{\ln b})^x = e^{\ln b \cdot x}$

and, $y' = e^{\ln b\, x} \cdot \frac{d}{dx} \ln b\, x$

$$= \boxed{b^x \cdot \ln b}$$

note if $b = e$, $y = e^x$

$y' = \ln e \cdot e^x = 1e^x$

as it should.

Example 8: Find the derivative of the following functions.

(a) $y = 5^x$

$$\boxed{y' = (\ln 5)\, 5^x}$$

(b) $f(x) = 10^{\cos x}$

$$f'(x) = \ln 10 \cdot 10^{\cos x} \cdot \frac{d}{dx}\cos x$$
$$\boxed{f'(x) = -(\ln 10)\, 10^{\cos x} \sin x}$$

(c) $g(x) = e^{-2x^2}$

$$g'(x) = e^{-2x^2} \cdot \frac{d}{dx}(-2x^2)$$
$$= \boxed{-4x e^{-2x^2}}$$

Example 9: Find the derivative of the following functions.

(a) $f(x) = 5^{3^{x^2}}$

$$f'(x) = \ln 5 \cdot 5^{3^{x^2}} \cdot \frac{d}{dx} 3^{x^2}$$

$$= \ln 5 \cdot 5^{3^{x^2}} \cdot \ln 3 \cdot 3^{x^2} \cdot \frac{d}{dx} x^2$$

$$= \boxed{2x \ln 5 \cdot \ln 3 \cdot 5^{3^{x^2}} 3^{x^2}}$$

(b) $y = \sin\sqrt{\cos(\cot(3x))}$

$$y' = \cos\sqrt{\cos(\cot 3x)} \cdot \frac{d}{dx}\sqrt{\cos\cot(3x)}$$

$$= \cos\sqrt{\cos(\cot 3x)} \cdot \frac{1}{2}(\cos(\cot(3x)))^{-1/2} \cdot \frac{d}{dx}\cos(\cot 3x)$$

$$= \frac{\cos\sqrt{\cos(\cot 3x)}}{2\sqrt{\cos(\cot 3x)}} \cdot (-\sin(\cot 3x)) \cdot \frac{d}{dx}\cot 3x$$

$$= \boxed{\frac{3\cos\sqrt{\cos(\cot 3x)}\,\sin(\cot 3x)\csc^2(3x)}{2\sqrt{\cos(\cot 3x)}}}$$

Example 10: Find the points on the graph of the function $f(x) = 2\cos x + \cos^2 x$ at which the tangent is horizontal.

$$f'(x) = -2\sin x + 2\cos x \cdot \frac{d}{dx}\cos x$$

$$= -2\sin x + 2\cos x(-\sin x)$$

$$0 = -2\sin x(1 + \cos x)$$

$0 = \sin x$ $\qquad$ $1 + \cos x = 0$

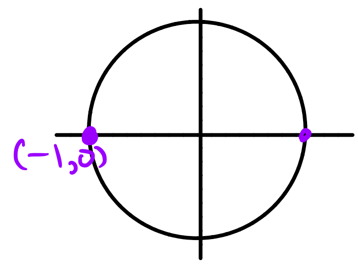
$\qquad\qquad\qquad\quad \cos x = -1$

$x = 0, \pm\pi, \pm 2\pi \ldots$ $\qquad$ $x = \pi(2n+1)$

$\uparrow$ gives all odds

these are in here

$\boxed{x = \pi n,\ n \text{ an integer}}$



Example 11: Find the 100th derivative of $y = \sin(5x)$

$y = \sin(5x)$

$y' = 5\cos(5x)$

$y'' = -5^2\sin(5x)$

$y''' = -5^3\cos(5x)$

$y'''' = 5^4\sin(5x)$

$100 \div 4 = 25$

end up here!

$$\boxed{y^{(100)} = 5^{100}\sin(5x)}$$

Example 12: The average BAC of eight male subjects was measured after consumption of 15 mL of ethanol. The resulting data were modeled by the concentration function

$$C(t) = 0.0225te^{-.0467t} \qquad \text{product + chain}$$

where t is measured in minutes after consumption and C is measured in mg/mL.

(a) How rapidly was BAC increasing after 10 minutes?

$$C'(t) = 0.0225e^{-.0467t} + 0.0225t(-.0467)e^{-.0467t}$$
$$= 0.0225e^{-.0467t}(1 - 0.0467t)$$
$$C'(10) = 0.0225e^{-.467}(1 - 0.467)$$
$$\approx \boxed{0.00755 \text{ (mg/mL)/min}}$$

(b) How rapidly was BAC decreasing half an hour later?

$$C'(30) = 0.0225e^{-0.0467(30)}(1 - 0.0467(30))$$
$$\approx \boxed{-0.00223 \text{ (mg/mL)/min}}$$

Example 13: A model for the length of day (in hours) in Philadelphia on the t-th day of the year is

$$L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t - 80)\right].$$

Use this model to compare the number of hours of daylight is increasing in Philadelphia on January 15th ($t = 15$) and March 21st ($t = 80$).

$$L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\cdot\frac{2\pi}{365} = \frac{5.6\pi}{365}\cos\left(\frac{2\pi}{365}(t-80)\right)$$

$$L'(15) = \frac{5.6\pi}{365}\cos\left(\frac{-130\pi}{365}\right) \approx \boxed{0.021 \text{ hrs/day}}$$
$$\text{or} \approx \boxed{1.263 \text{ min/day}}$$

$$L'(80) = \frac{5.6\pi}{365}\cos(0) \approx \boxed{0.048 \text{ hrs/day}}$$
$$\text{or} \approx \boxed{2.892 \text{ min/day}}$$

Example 14: Use the product rule and chain rule to prove the quotient rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left[f(x)\cdot(g(x))^{-1}\right]$$
$$= f'(x)[g(x)]^{-1} + f(x)(-1)[g(x)]^{-2}g'(x)$$
$$= \frac{f'(x)}{g(x)}\frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}$$
$$= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

Example 15: Find the derivatives of the following functions.

(a) $y = \cos^2(\cot(2x)) = (\cos(\cot(2x)))^2$

$$y' = 2\cos(\cot(2x))\cdot\frac{d}{dx}\cos(\cot 2x)$$
$$= 2\cos(\cot(2x))(-\sin(\cot(2x)))\cdot\frac{d}{dx}\cot(2x)$$
$$= 2\cos(\cot(2x))(-\sin(\cot(2x))\cdot(-\csc^2(2x))\cdot 2$$
$$= \boxed{4\cos(\cot(2x))\sin(\cot(2x))\csc^2(2x)}$$

(b) $y = x^3e^{-1/x^2}$

$$y' = 3x^2e^{-1/x^2} + x^3e^{-1/x^2}\cdot\frac{d}{dx}(-x^{-2})$$
$$= 3x^2e^{-1/x^2} + x^3e^{-1/x^2}(-1(-2)x^{-3})$$
$$= 3x^2e^{-1/x^2} + x^3e^{-1/x^2}\cdot 2/x^3$$
$$= 3x^2e^{-1/x^2} + 2e^{-1/x^2}$$
$$= \boxed{e^{-1/x^2}(3x^2+2)}$$

Example 16: Find an equation of the tangent line to the curve $y = 3^{\sin x}$ at the point where $x = 0$.

① find $y' = \ln 3\cdot 3^{\sin x}\cdot\frac{d}{dx}\sin x$

$= \ln 3\cdot 3^{\sin x}\cos x$

② find $m = y'(0) = \ln 3\ 3^{\sin 0}\cos 0$

$m = \ln 3$

③ find the point: $x=0$, $y = 3^{\sin 0} = 3^0 = 1$

④ equation $y - y_1 = m(x - x_1)$

$y - 1 = \ln 3(x - 0)$

$\boxed{y = (\ln 3)x + 1}$